MATH 205: Homework 5

Due Fri Nov 10

Most of these problems are from "Multivariable Mathematics" by T. Shifrin. I recommend at least thinking about all the exercises, even if they are not assigned, as you read through the textbook. I also recommend trying to prove all the Theorems which are left unproven in the book.

Problem 1. Suppose that $\omega \in \Lambda^k(\mathbb{R}^n)^*$ for k odd, show that $\omega \wedge \omega = 0$. Give an example to show that if k even then $\omega \wedge \omega$ may be nonzero.

Problem 2. Compute the exterior derivatives of the following differential forms:

(1) $\omega = e^{xy}dx$
(2) $\omega = z^2dx + x^2dy + y^2dz$
(3) $\omega = x^2dy \wedge dz + y^2dz \wedge dx + z^2dx \wedge dy$

Problem 3. Can the given 1-form ω be written as $\omega = df$ for a function $f : \mathbb{R}^n \to \mathbb{R}$? If so find f.

(1) $\omega = -ydx + xdy$
(2) $\omega = 2xydx + x^2dy$
(3) $\omega = ydx + zdy + xdz$
(4) $\omega = \frac{x}{x^2+y^2}dx + \frac{y}{x^2+y^2}dy$

Problem 4. Compute the pullback $g^*\omega$, simplifying as much as possible.

(1) $g : \mathbb{R} \to \mathbb{R}^2$, $g(t) = (3\cos(2t), 3\sin(2t))$, $\omega = -ydx + xdy$
(2) $g : \mathbb{R}^2 \to \mathbb{R}^2$, $g(r, \theta) = (3r\cos(2\theta), 3r\sin(2\theta))$, $\omega = -ydx + xdy$
(3) $g : \mathbb{R}^2 \to \mathbb{R}^3$, $g(u, v) = (\cos u, \sin u, v)$, $\omega = zdx + xdy + ydz$
(4) $g : \mathbb{R}^2 \to \mathbb{R}^3$, $g(u, v) = (\cos u, \sin u, v)$, $\omega = zdx \wedge dy + ydz \wedge dx$

Problem 5. Can every $a \in \Lambda^k(\mathbb{R}^n)$ be written as $v_1 \wedge \cdots \wedge v_k$ for some vectors $v_1, \ldots, v_k \in \mathbb{R}^n$? (Recall that the space $\Lambda^k(\mathbb{R}^n)$ is defined as the linear span of $\{v_1 \wedge \cdots \wedge v_k : v_1, \ldots, v_k \in \mathbb{R}^n\}$) (Hint: Show that $a \wedge a$ may be non-zero)

Problem 6. Let $g : (0, \infty) \times (0, 2\pi) \times (0, \pi) \to \mathbb{R}^3$ be the spherical coordinate mapping $g(\rho, \theta, \phi) = (\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\theta)$, compute $g^*\omega$ when $\omega = dx \wedge dy \wedge dz$.

Problem 7. Suppose that $\phi_1, \ldots, \phi_k \in (\mathbb{R}^n)^*$ and $v_1, \ldots, v_k \in \mathbb{R}^n$, show that

$$\phi_1 \wedge \cdots \wedge \phi_k(v_1, \ldots, v_k) = \det[\phi_i(v_j)].$$

(Hint: take v_j to be the standard basis vectors e_j, and expand $\phi_i = \sum a_{ij}dx_j$, then compute both sides)

Problem 8. Let C be an oriented curve in $\mathbb{R}^2$ and let n the unit outward normal (i.e. $\{n, T\}$ is a right handed basis for $\mathbb{R}^2$, where T is the tangent to C). Let $F = (F_1, F_2)$ be a vector field in $\mathbb{R}^2$ and $\omega = F_1dx + F_2dy$ be the corresponding 1-form. Show that

$$\int_C F \cdot n \; ds = \int_C F_1dy - F_2dx.$$

Here $\int_C f \; ds$ is the arc-length integral defined by $\int_a^b f(\gamma(t))|\gamma'(t)|dt$ if γ parametrizes C. The quantity on the left is called the *flux* of the vector field F through C. Conclude that if $C = \partial\Omega$ then,

$$\int_C F \cdot nds = \int_\Omega \nabla \cdot F = \int_\Omega \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}.$$

Problem 9. An ant finds himself in the xy-plane in the presence of the force field $F = (y^3 + x^2y, 2x^2 - 6xy)$. Around what simple closed curve should he travel counter-clockwise inorder to maximize the work done on him by F? (Hint: use Green's theorem)