MATH 275: Homework 7
Due Thursday, May 19

Problem 1. Consider the following scalar conservation law

$$\begin{cases} u_t + f(u)_x = 0 & \text{in} \quad \mathbb{R} \times (0, \infty) \\ u(x, 0) = -x & \text{in} \quad \mathbb{R}. \end{cases} \tag{0.1}$$

Show that if $f''(z) \geq \theta$ then $\inf_{y \in \mathbb{R}} u_x(y) \to -\infty$ in a finite time.

Problem 2. Consider the solution of Burger's equation with smooth initial data u_0,

$$\begin{cases} u_t + u u_x = 0 & \text{in} \quad \mathbb{R} \times (0, \infty) \\ u(x, 0) = u_0(x) & \text{in} \quad \mathbb{R}. \end{cases} \tag{0.2}$$

Suppose that u_0' is bounded in $\mathbb{R}$.

(a) Compute, in terms of u_0, the first time t_* (possibly $+\infty$) when characteristics cross.
(b) Derive a necessary and sufficient condition on u_0 so that $t_* = +\infty$ and therefore (0.2) has a smooth solution for all $t > 0$.

Problem 3. (Evans 2nd edition, Chapter 3 problem 19) Assume that $f(0) = 0$ and u is a continuous integral solution of

$$\begin{cases} u_t + f(u)_x = 0 & \text{in} \quad \mathbb{R} \times (0, \infty) \\ u(x, 0) = u_0(x) & \text{in} \quad \mathbb{R}, \end{cases} \tag{0.3}$$

and u has compact support in $\mathbb{R} \times [0, T]$ for each $T > 0$, prove that,

$$\int_{\mathbb{R}} u(x, t)\ dx = \int_{\mathbb{R}} u_0(x)\ dx$$

for all $t > 0$.

Problem 4. Consider the transport equation

$$u_t + a(x, t) \cdot \nabla u = 0.$$

Prove that if u is a solution and $\beta : \mathbb{R} \to \mathbb{R}$ is an arbitrary C^1 function then $\beta(u(x, t))$ is a solution as well.

Problem 5. Find the characteristics for the problem,

$$u_t = \frac{1}{2}((u_x)^2 + x^2) \tag{0.4}$$

with the initial condition $u(x, 0) = x$. The solution will not be defined for $|t| \geq \pi/2$. Explain that from the behavior of the characteristics.