Higher order Derivatives

Let $f : E \text{ open } \subseteq \mathbb{R}^n \to \mathbb{R}$

w/ partial derivatives $D_1 f, \ldots, D_n f$

if $D_j f$ are themselves diff then we can define the second order partials

$$D_{ij}^2 f = D_i D_j f \qquad i, j = 1, \ldots, n$$

If all second partials are cts we say

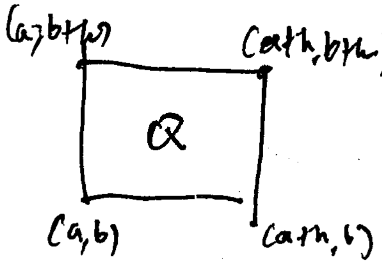
f is C^2.

vector valued mappings are C^2 if each component is C^2.

It is possible that $D_{ij}^2 f \neq D_{ji}^2 f$ even w/ both derivatives existing, but if $D_{ij}^2 f$ are all cts this cannot happen

Thm f defined on E open $\subset \mathbb{R}^2$ and $D_1 f$, $D_{21}^2 f$ exist in E. If Q is an closed axis parallel rectangle in E and

$$\Delta(f,Q) = f(a+h,b+k) + f(a,b) - f(a+h,b) - f(a,b+k)$$



then $\exists$ $(x,y) \in Q$ w/

$$\Delta(f,Q) = hk\, D_{21}^2 f(x,y)$$

proof: Call $u(t) = f(t,b+k) - f(t,b)$

then

$$\begin{aligned} \Delta(f,Q) &= u(a+h) - u(h) \\ &= h u'(x) \\ &= h[D_1 f(x,b+k) - D_1 f(x,b)] \\ &= hk\, D_{21}^2 f(x,y) \end{aligned}$$

by applying MVT twice.

Thm ~~Suppose $f : E$ open $\subset \mathbb{R}^n \Rightarrow$~~

Suppose f defined on E open in $\mathbb{R}^2$

and that $D_1 f$, $D_{21}^2 f$ and $D_2 f$

exist in E and $D_{21}^2 f$ is cts at $(a,b) \in E$

Then $D_{12}^2 f$ exists at (a,b) ~~in E~~ and

$$D_{12}^2 f(a,b) = D_{21}^2 f(a,b)$$

Cor: $D_{12}^2 f = D_{21}^2 f$ if f is C^2.

proof: put Q the rectangle as before,

let $\varepsilon > 0$ if h, k small enough

$$(x,y) \in Q \Rightarrow |D_{21}^2 f(x,y) - D_{21}^2 f(a,b)| < \varepsilon$$

using the previous "MVT" type result

$$\left| \frac{\Delta(f,Q)}{hk} - D_{21}^2 f(a,b) \right| < \varepsilon$$

fix $h \neq 0$ and let $k \to 0$ to get

$$\left| \frac{1}{h}\left[D_2 f(a+h,b) - D_2 f(a,b) \right] - D_{21}^2 f(a,b) \right| < \varepsilon$$

Since ε arbitrary $\Rightarrow$ $\lim_{h \to 0} \frac{D_2 f(a+h,b) - D_2 f(a,b)}{h}$ exists and $= D_{21}^2 f(a,b)$